Subject Name: **Strength of materials**

Subject Code: **CE-3003**

Semester: **3rd**

UNIT-IV

Column & Struts

Columns and Struts: Theory of columns, Slenderness ratio, Direct and bending stresses in short columns, Kern of a section. Buckling and stability, Euler's buckling/crippling load for columns with different end conditions, Rankin's formula, Eccentric loads and the Secant formula-Imperfections in columns.
Thin Pressure Vessels: cylinders and spheres. Stress due to internal pressure, Change in diameter and volume.
Theories of failure.

Introduction:

Structural members which carry compressive loads may be divided into two broad categories depending on their relative lengths and cross-sectional dimensions.

Columns:

Short, thick members are generally termed columns and these usually fail by crushing when the yield stress of the material in compression is exceeded.

Struts:

Long, slender columns are generally termed as struts, they fail by buckling some time before the yield stress in compression is reached. The buckling occurs owing to one the following reasons.

(a). the strut may not be perfectly straight initially.

(b). the load may not be applied exactly along the axis of the Strut.

(c). one part of the material may yield in compression more readily than others owing to some lack of uniformity in the material properties through out the strut.

In all the problems considered so far we have assumed that the deformation to be both progressive with increasing load and simple in form i.e. we assumed that a member in simple tension or compression becomes progressively longer or shorter but remains straight. Under some circumstances however, our assumptions of progressive and simple deformation may no longer hold good and the member become unstable. The term strut and column are widely used, often interchangeably in the context of buckling of slender members.]

At values of load below the buckling load a strut will be in stable equilibrium where the displacement caused by any lateral disturbance will be totally recovered when the disturbance is removed. At the buckling load the strut is said to be in a state of neutral equilibrium, and theoretically it should than be possible to gently deflect the strut into a simple sine wave provided that the amplitude of wave is kept small.

Theoretically, it is possible for struts to achieve a condition of unstable equilibrium with loads exceeding the buckling load, any slight lateral disturbance then causing failure by buckling, this condition is never achieved in practice under static load conditions. Buckling occurs immediately at the point where the buckling load is reached, owing to the reasons stated earlier.

The resistance of any member to bending is determined by its flexural rigidity EI and is The quantity I may be written as I = Ak^2,

Where I = area of moment of inertia

A = area of the cross-section

k = radius of gyration.

The load per unit area which the member can withstand is therefore related to k. There will be two principal moments of inertia, if the least of these is taken then the ratio

$$\frac{l}{k} \quad \text{i.e.} \quad \frac{\text{length of member}}{\text{least radius of gyration}}$$

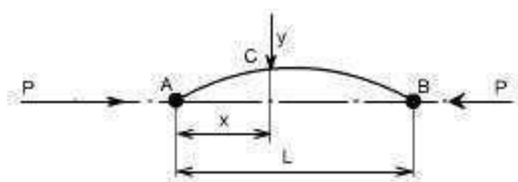
Is called the slenderness ratio. It's numerical value indicates whether the member falls into the class of columns or struts.

Euler's Theory: The struts which fail by buckling can be analyzed by Euler's theory. In the following sections, different cases of the struts have been analyzed.

Case A: Strut with pinned ends:

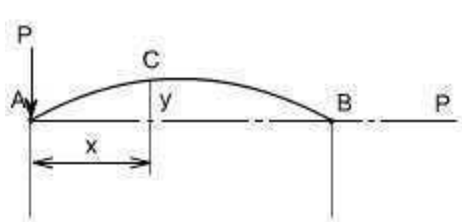
Consider an axially loaded strut, shown below, and is subjected to an axial load ‘P' this load ‘P' produces a deflection ‘y' at a distance ‘x' from one end.

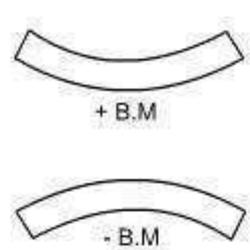
Assume that the ends are either pin jointed or rounded so that there is no moment at either end.



Assumption:

The strut is assumed to be initially straight, the end load being applied axially through centroid.





According to sign convention

$$B.M|_C = -Py$$

Futher,we know that

$$EI\frac{d^2y}{dx^2} = M$$

$$EI\frac{d^2y}{dx^2} = -P.y = M$$

In this equation 'M' is not a function 'x'. Therefore this equation can not be integrated directly as has been done in the case of deflection of beams by integration method.

Thus,

$$EI\frac{d^2y}{dx^2} + Py = 0$$

Though this equation is in 'y' but we can't say at this stage where the deflection would be maximum or minimum.

So the above differential equation can be arranged in the following form $\frac{d^2y}{dx^2} + \frac{Py}{EI} = 0$

Let us define a operator

$$D = d/dx$$

$$(D^2 + n^2)\, y = 0 \text{ where } n^2 = P/EI$$

This is a second order differential equation which has a solution of the form consisting of complimentary function and particular integral but for the time being we are interested in the complementary solution only[in this P.I = 0; since the R.H.S of Diff. equation = 0]

Thus

$$y = A\cos(nx) + B\sin(nx)$$

Where A and B are some constants. Therefore

$$y = A\cos\sqrt{\frac{P}{EI}}\,x + B\sin\sqrt{\frac{P}{EI}}\,x$$

In order to evaluate the constants A and B let us apply the boundary conditions,

(i) at $x = 0;\ y = 0$

(ii) at $x = L;\ y = 0$

Applying the first boundary condition yields A = 0.

Applying the second boundary condition gives

$$B\sin\left(L\sqrt{\frac{P}{EI}}\right) = 0$$

$$\text{Thus either } B = 0, \text{or } \sin\left(L\sqrt{\frac{P}{EI}}\right) = 0$$

if B=0,that y0 for all values of x hence the strut has not buckled yet.Therefore,the solution required is

$$\sin\left(L\sqrt{\frac{P}{EI}}\right) = 0 \quad \text{or} \quad \left(L\sqrt{\frac{P}{EI}}\right) = \pi \text{ or } nL = \pi$$

$$\text{or } \sqrt{\frac{P}{EI}} = \frac{\pi}{L} \quad \text{or} \quad P = \frac{\pi^2 EI}{L^2}$$

From the above relationship the least value of P which will cause the strut to buckle, and it is called the " **Euler Crippling Load** " P_e from which w obtain.

$$\boxed{P_e = \frac{\pi^2 EI}{L^2}}$$

It may be noted that the value of I used in this expression is the least moment of inertia

It should be noted that the other solutions exists for the equation

$$\sin\left(l\sqrt{\frac{P}{EI}}\right) = 0 \qquad \text{i.e. } \sin nL = 0$$

The interpretation of the above analysis is that for all the values of the load P, other than those which make sin nL = 0; the strut will remain perfectly straight since

y = B sin nL = 0

For the particular value of

$$P_e = \frac{\pi^2 EI}{L^2}$$

$$\sin nL = 0 \quad \text{or } nL = \pi$$

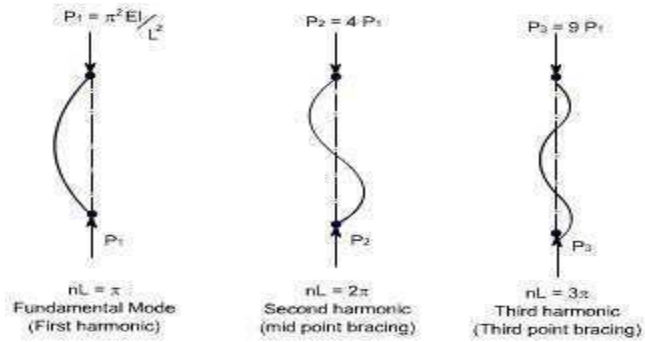
$$\text{Therefore } n = \frac{\pi}{L}$$

$$\text{Hence } y = B \sin nx = B \sin \frac{\pi x}{L}$$

Then we say that the strut is in a state of neutral equilibrium, and theoretically any deflection which it suffers will be maintained. This is subjected to the limitation that 'L' remains sensibly constant and in practice slight increase in load at the critical value will cause the deflection to increase appreciably until the material fails by yielding.

The solution chosen of nL = p is just one particular solution; the solutions nL= 2p, 3p, 5p etc are equally valid mathematically and they do, infact, produce values of 'P_e' which are equally valid for modes of buckling of strut different from that of a simple bow. Theoretically therefore, there are an infinite number of values of P_e , each corresponding with a different mode of buckling.

The solution nL = 2p produces buckling in two half – waves, 3p in three half-waves etc.



$$L\sqrt{\frac{P}{EI}} = \pi \quad \text{or} \quad P_1 = \frac{\pi^2 EI}{L^2}$$

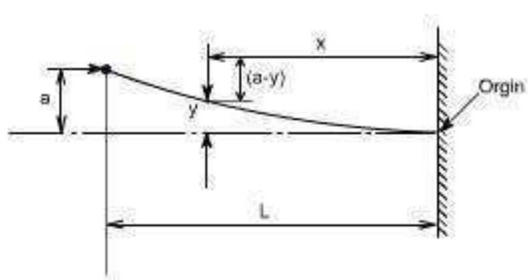
$$\text{If } L\sqrt{\frac{P}{EI}} = 2\pi \text{ or } P_2 = \frac{4\pi^2 EI}{L^2} = 4P_1$$

$$\text{If } L\sqrt{\frac{P}{EI}} = 3\pi \text{ or } P_3 = \frac{9\pi^2 EI}{L^2} = 9P_1$$

If load is applied sufficiently quickly to the strut, then it is possible to pass through the fundamental mode and to achieve at least one of the other modes which are theoretically possible. In practical loading situations, however, this is rarely achieved since the high stress associated with the first critical condition generally ensures immediate collapse.

<u>struts and columns with other end conditions:</u> Let us consider the struts and columns having different end conditions

Case b: One end fixed and the other free:



writing down the value of bending moment at the point C

$$B.M|_c = P(a - y)$$

Hence, the differential equation becomes,

$$EI\frac{d^2y}{dx^2} = P(a - y)$$

On rearranging we get

$$\frac{d^2y}{dx^2} + \frac{Py}{EI} = \frac{Pa}{EI}$$

Let $\frac{P}{EI} = n^2$

Hence in operator form, the differential equation reduces to $(D^2 + n^2) y = n^2a$

The solution of the above equation would consist of complementary solution and particular solution, therefore

$$y_{gen} = A\cos(nx) + \sin(nx) + P.I$$

where

P.I = the P.I is a particular value of y which satisfies the differential equation

Hence $y_{P.I} = a$

Therefore the complete solution becomes

$$Y = A\cos(nx) + B\sin(nx) + a$$

Now imposing the boundary conditions to evaluate the constants A and B

(i) at $x = 0$; $y = 0$

This yields $A = -a$

(ii) at $x = 0$; $dy/dx = 0$

This yields $B = 0$

Hence

$$y = -a\cos(nx) + a$$

Futher, at $x = L$; $y = a$

Therefore

$$a = -a\cos(nx) + a \quad \text{or } 0 = \cos(nL)$$

Now the fundamental mode of buckling in this case would be

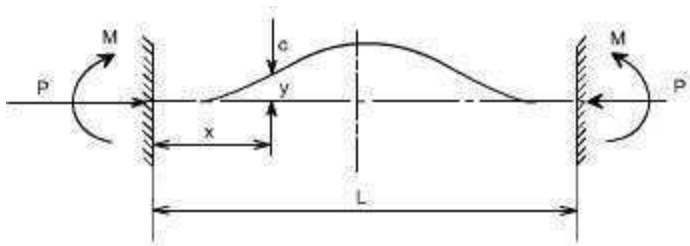
$$nL = \frac{\pi}{2}$$

$$\sqrt{\frac{P}{EI}}\,L = \frac{\pi}{2}, \text{Therefore, the Euler's crippling load is given as}$$

$$P_e = \frac{\pi^2 EI}{4L^2}$$

Case 3

Strut with fixed ends:



Due to the fixed end supports bending moment would also appears at the supports, since this is the property of the support.

$$EI\frac{d^2y}{dx^2} = M - Py$$

or $\frac{d^2y}{dx^2} + \frac{P}{EI} = \frac{M}{EI}$

$n^2 = \frac{P}{EI}$, Therefore in the operator from, the equation reduces to

$$\left(D^2 + n^2\right) y = \frac{M}{EI}$$

$$y_{general} = y_{complementary} + y_{particular\ integral}$$

$$y|_{P.I} = \frac{M}{n^2 EI} = \frac{M}{P}$$

Hence the general solution would be

$$y = B\ Cosnx + A\ Sinnx + \frac{M}{P}$$

Boundry conditions relevant to this case are at x=0;y=0

$$B = -\frac{M}{P}$$

Also at $x = 0; \frac{dy}{dx} = 0$ hence

A=0

Therefore,

$$y = -\frac{M}{P} Cos\ nx + \frac{M}{P}$$

$$y = \frac{M}{P}\left(1 - Cos\ nx\right)$$

Futher, it may be noted that at $x = L; y = 0$

Then $0 = \frac{M}{P}\left(1 - Cos\ nL\right)$

Thus, either $\frac{M}{P} = 0$ or $\left(1 - Cos\ nL\right) = 0$

obviously, $\left(1 - Cos\ nL\right) = 0$

$cos\ nL = 1$

Hence the least solution would be

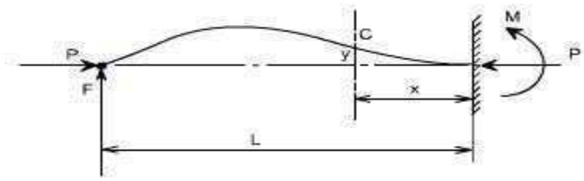
$nL = 2\pi$

$\sqrt{\frac{P}{EI}}\ L = 2\pi$, Thus, the buckling load or crippling load is

$$P_e = \frac{4\pi^2.EI}{L^2}$$

Bending Moment at point C = M – P.y

Case 4 <u>One end fixed, the other pinned</u>



In order to maintain the pin-joint on the horizontal axis of the unloaded strut, it is necessary in this case to introduce a vertical load F at the pin. The moment of F about the built in end then balances the fixing moment.

With the origin at the built in end, the B,M at C is given as

$$EI\frac{d^2y}{dx^2} = -Py + F(L-x)$$

$$EI\frac{d^2y}{dx^2} + Py = F(L-x)$$

Hence

$$\frac{d^2y}{dx^2} + \frac{P}{EI}y = \frac{F}{EI}(L-x)$$

In the operator form the equation reduces to

$$\left(D^2 + n^2\right)y = \frac{F}{EI}(L-x)$$

$$y_{particular} = \frac{F}{n^2EI}(L-x) \text{ or } y = \frac{F}{P}(L-x)$$

The full solution is therefore

$$y = A\,\text{Cos}\,mx + B\,\text{Sin}\,nx + \frac{F}{P}(L-x)$$

The boundry conditions relevants to the problem are at x=0;y=0

$$\text{Hence } A = -\frac{FL}{P}$$

$$\text{Also at } x = 0; \frac{dy}{dx} = 0$$

$$\text{Hence } B = \frac{F}{nP}$$

$$\text{or } y = -\frac{FL}{P}\text{Cos}\,nx + \frac{F}{nP}\text{Sin}\,nx + \frac{F}{P}(L-x)$$

$$y = \frac{F}{nP}\left[\text{Sin}\,nx - nL\,\text{Cos}\,nx + n(L-x)\right]$$

Also when x = L ; y = 0

Therefore

$$nL\ \text{Cos}\ nL = \text{Sin}\ nL \quad \text{or } \tan nL = nL$$

The lowest value of nL (neglecting zero) which satisfies this condition and which therefore produces the fundamental buckling condition is nL = 4.49radian

$$\text{or} \sqrt{\frac{P}{EI}}\, L = 4.49$$

$$\frac{P_e}{EI} L^2 = 20.2$$

$$P_e = \frac{2.05\pi^2\, EI}{L^2}$$

Equivalent Strut Length:

Having derived the results for the buckling load of a strut with pinned ends the Euler loads for other end conditions may all be written in the same form.

$$\text{i.e. } P_e = \frac{\pi^2 EI}{L^2}$$

Where L is the equivalent length of the strut and can be related to the actual length of the strut depending on the end conditions.

For case(c) see the figure, the column or strut has inflection points at quarter points of its unsupported length. Since the bending moment is zero at a point of inflection, the freebody diagram would indicates that the middle half of the fixed ended is equivalent to a hinged column having an effective length $L_e = L / 2$.

The four different cases which we have considered so far are:

(a) Both ends pinned (c) One end fixed, other free

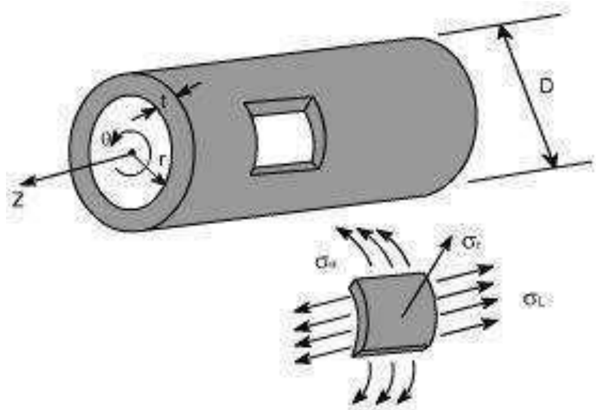
(b) Both ends fixed (d) One end fixed and other pinned

Pressurized thin walled cylinder:

In the analysis of this walled cylinders subjected to internal pressures it is assumed that the radial plans remains radial and the wall thickness does not change due to internal pressure. Although the internal pressure acting on the wall causes a local compressive stresses (equal to pressure) but its value is neglibly small as compared to other stresses & hence the set of stress of an element of a thin walled pressure is considered a biaxial one.

Further in the analysis of them walled cylinders, the weight of the fluid is considered neglibly.

Let us consider a long cylinder of circular cross - section with an internal radius of R 2 and a constant wall thickness' t' as showing fig.



This cylinder is subjected to a difference of hydrostatic pressure of ‘p' between its inner and outer surfaces. In many cases, ‘p' between gage pressure within the cylinder, taking outside pressure to be ambient.

By thin walled cylinder we mean that the thickness‘t' is very much smaller than the radius Ri and we may quantify this by stating than the ratio t / Ri of thickness of radius should be less than 0.1.

Type of failure: Such a component fails in since when subjected to an excessively high internal pressure. While it might fail by bursting along a path following the circumference of the cylinder. Under normal circumstance it fails by circumstances it fails by bursting along a path parallel to the axis. This suggests that the hoop stress is significantly higher than the axial stress.

Applications:

Liquid storage tanks and containers, water pipes, boilers, submarine hulls, and certain air plane components are common examples of thin walled cylinders and spheres, roof domes.

ANALYSIS: In order to analyze the thin walled cylinders, let us make the following assumptions :

- There are no shear stresses acting in the wall.
- The longitudinal and hoop stresses do not vary through the wall.
- Radial stresses ?r which acts normal to the curved plane of the isolated element are neglibly small as compared to other two stresses especially when $\left[\frac{t}{R_i} < \frac{1}{20}\right]$

Thin Cylinders Subjected to Internal Pressure:

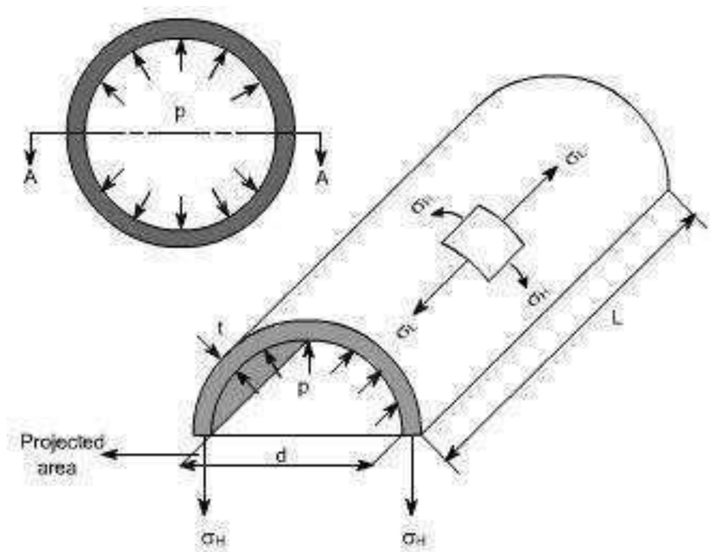
When a thin – walled cylinder is subjected to internal pressure, three mutually perpendicular principal stresses will be set up in the cylinder materials, namely

- Circumferential or hoop stress
- The radial stress
- Longitudinal stress

Now let us define these stresses and determine the expressions for them

Hoop or circumferential stress:

This is the stress which is set up in resisting the bursting effect of the applied pressure and can be most conveniently treated by considering the equilibrium of the cylinder.



In the figure we have shown a one half of the cylinder. This cylinder is subjected to an internal pressure p.

i.e.

p = internal pressure d = inside diameter

L = Length of the cylinder

t = thickness of the wall

Total force on one half of the cylinder owing to the internal pressure 'p'

= p x Projected Area

= p x d x L

= **p .d. L** ------- (1)

The total resisting force owing to hoop stresses σ_H set up in the cylinder walls

= **2 .** σ_H**.L.t** ---------(2)

Because σ_H.L.t. is the force in the one wall of the half cylinder. the equations (1) & (2) we get

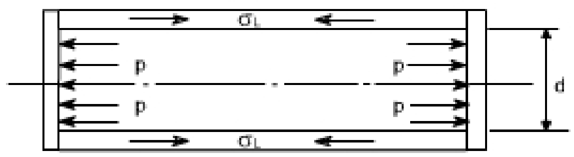
2 . σ_H**. L . t = p . d . L**

$$\sigma_H = (p \cdot d) / 2t$$

$$\boxed{\text{Circumferential or hoop Stress } (\sigma_H) = (p \cdot d)/2t}$$

Longitudinal Stress:

Consider now again the same figure and the vessel could be considered to have closed ends and contains a fluid under a gage pressure p. Then the walls of the cylinder will have a longitudinal stress as well as a circumferential stress.



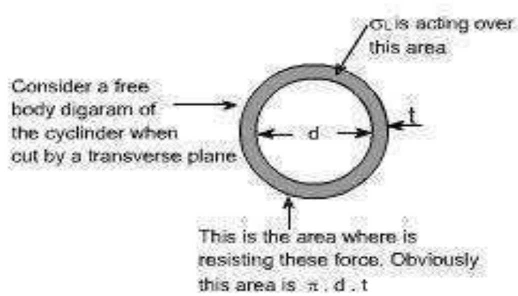
Total force on the end of the cylinder owing to internal pressure

= pressure x area

= p x $\pi d^2/4$

Area of metal resisting this force =πd.t. (approximately)

Because πd is the circumference and this is multiplied by the wall thickness



Hence the longitudnal stresses

$$\text{Set up} = \frac{\text{force}}{\text{area}} = \frac{[p \times \pi d^2/4]}{\pi d t}$$

$$= \frac{pd}{4t} \quad \text{or} \quad \sigma_L = \frac{pd}{4t}$$

or alternatively from equilibrium conditions

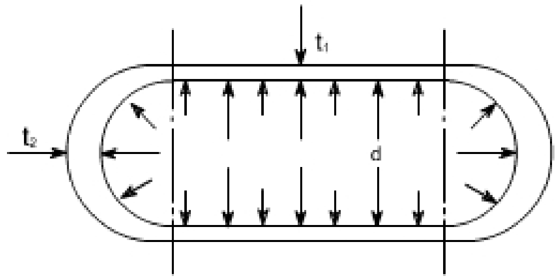
$$\sigma_L.(\pi d t) = p.\frac{\pi d^2}{4}$$

$$\text{Thus } \boxed{\sigma_L = \frac{pd}{4t}}$$

Cylindrical Vessel with Hemispherical Ends:

Let us now consider the vessel with hemispherical ends. The wall thickness of the cylindrical and

hemispherical portion is different. While the internal diameter of both the portions is assumed to be equal.
Let the cylindrical vassal is subjected to an internal pressure p.



For the Cylindrical Portion

hoop or circumferential stress $= \sigma_{HC}$ 'c' here synifies the cylindrical portion.

$$= \frac{pd}{2t_1}$$

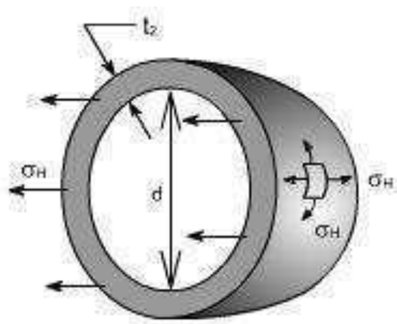
longitudnal stress $= \sigma_{LC}$

$$= \frac{pd}{4t_1}$$

hoop or circumferential strain $\in_2 = \frac{\sigma_{HC}}{E} - \nu\frac{\sigma_{LC}}{E} = \frac{pd}{4t_1E}[2-\nu]$

or $\boxed{\in_2 = \frac{pd}{4t_1E}[2-\nu]}$

<u>For The Hemispherical Ends:</u>



Because of the symmetry of the sphere the stresses set up owing to internal pressure will be two mutually perpendicular hoops or circumferential stresses of equal values. Again the radial stresses are neglected in comparison to the hoop stresses as with this cylinder having thickness to diameter less than1:20.
Consider the equilibrium of the half – sphere Force on half sphere owing to internal pressure

= pressure x projected Area

$= p. \pi d^2/4$

Resisting force = $\sigma_H . \pi.d.t_2$

$\therefore \quad p.\frac{\pi.d^2}{4} = \sigma_H . \pi d.t_2$

$\Rightarrow \sigma_H \text{(for sphere)} = \frac{pd}{4t_2}$

similarly the hoop strain $= \frac{1}{E}\left[\sigma_H - \nu.\sigma_H\right] = \frac{\sigma_H}{E}[1-\nu] = \frac{pd}{4t_2E}[1-\nu]$ or $\boxed{\epsilon_{2s} = \frac{pd}{4t_2E}[1-\nu]}$

Fig – shown the (by way of dotted lines) the tendency, for the cylindrical portion and the spherical ends to expand by a different amount under the action of internal pressure. So owing to difference in stress, the two portions (i.e. cylindrical and spherical ends) expand by a different amount. This incompatibly of deformations causes a local bending and sheering stresses in the neighborhood of the joint. Since there must be physical continuity between the ends and the cylindrical portion, for this reason, properly curved ends must be used for pressure vessels.
Thus equating the two strains in order that there shall be no distortion of the junction

$$\frac{pd}{4t_1E}[2-\nu] = \frac{pd}{4t_2E}[1-\nu] \quad \text{or} \quad \frac{t_2}{t_1} = \frac{1-\nu}{2-\nu}$$

But for general steel works $\nu = 0.3$, therefore, the thickness ratios becomes

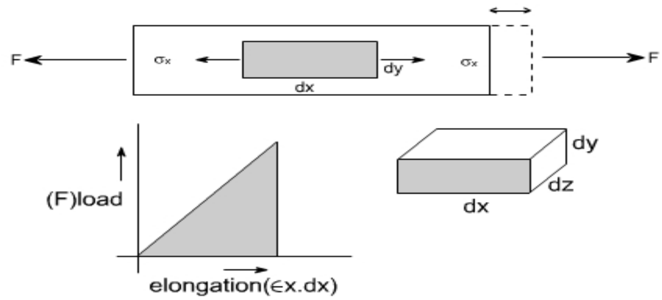
$$t_2 / t_1 = 0.7/1.7$$

i.e. the thickness of the cylinder walls must be approximately 2.4 times that of the hemispheroid ends for no distortion of the junction to occur.

THEORIES OF ELASTIC FAILURE

A number of theories have been proposed for the brittle and ductile materials.

Strain Energy: The concept of strain energy is of fundamental importance in applied mechanics. The application of the load produces strain in the bar. The effect of these strains is to increase the energy level of the bar itself. Hence a new quantity called strain energy is defined as the energy absorbed by the bar during the loading process. This strain energy is defined as the work done by load provided no energy is added or subtracted in the form of heat. Sometimes strain energy is referred to as internal work to distinguish it from external work 'W'. Consider a simple bar which is subjected to tensile force F, having a small element of dimensions dx, dy and dz.

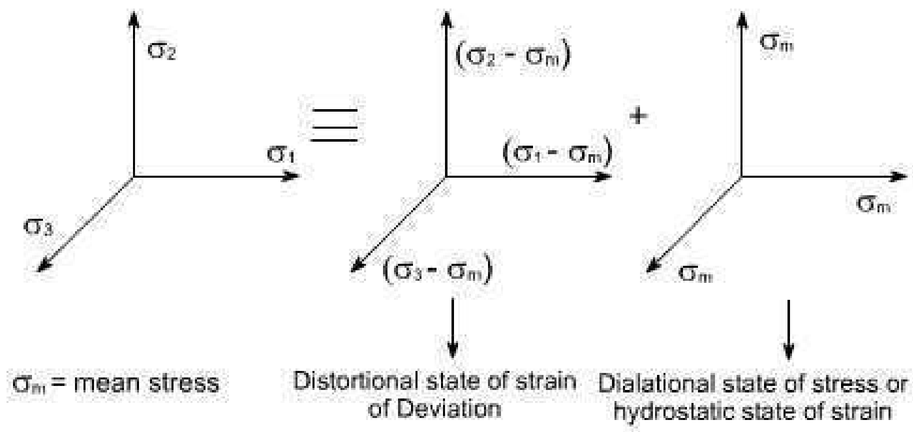


The strain energy U is the area covered under the triangle

$$U = \frac{1}{2} F . \in_x . d_x$$
$$= \frac{1}{2} \sigma_x . dydz . dx \in_x$$
$$= \frac{1}{2} \sigma_x . \in_x . dxdydz$$
$$= \frac{1}{2} \sigma_x \left(\frac{\sigma_x}{E} \right) . dxdydz$$
$$\boxed{\frac{U}{\text{volume}} = \frac{1}{2} \frac{\sigma_x^{2}}{E}}$$

A three dimension state of stress represented by σ_1, σ_2 and σ_3 may be thought of consisting of two distinct state of stresses
i.e Distortional state of stress
Deviator state of stress and dilatational state of stress

Hydrostatic state of stresses.



$$\sigma_m = \frac{(\sigma_1 + \sigma_2 + \sigma_3)}{3}$$

Thus, The energy which is stored within a material when the material is deformed is termed as a strain energy. The total strain energy U_r

$$U_T = U_d + U_H$$

U_d is the strain energy due to the Deviator state of stress and U_H is the strain energy due to the Hydrostatic state of stress. Further, it may be noted that the hydrostatic state of stress results in change of volume whereas the deviatory state of stress results in change of shape.

Different Theories of Failure: These are five different theories of failures which are generally used

(a) Maximum Principal stress theory (due to Rankine)

(b) Maximum shear stress theory (Guest - Tresca)

(c) Maximum Principal strain (Saint - Venant) Theory

(d) Total strain energy per unit volume (Haigh) Theory

(e) Shear strain energy per unit volume Theory (Von Mises & Hencky)

In all these theories we shall assume.

σ_{Yp} = stress at the yield point in the simple tensile test.

σ_1, σ_2, σ_3 the three principal stresses in the three dimensional complex state of stress systems in order of magnitude.

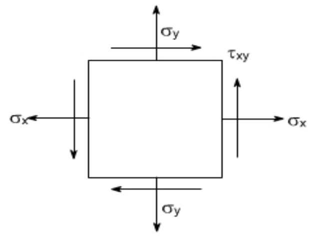
(a) Maximum Principal stress theory :

This theory assume that when the maximum principal stress in a complex stress system reaches the elastic limit stress in a simple tension, failure will occur.

Therefore the criterion for failure would be

$$\sigma_1 = \sigma_{yp}$$

For a two dimensional complex stress system σ_1 is expressed as



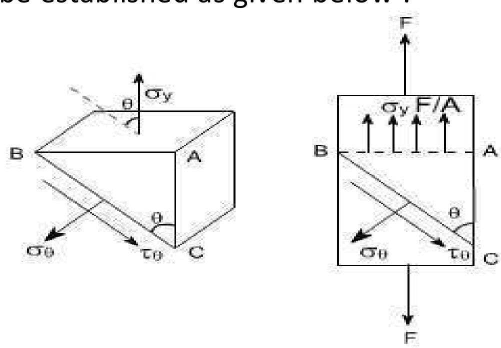
$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4.\tau_{xy}^2}$$
$$= \sigma_{yp}$$

Where σ_x, σ_y and τ_{xy} are the stresses in the any given complex stress system.

(b) Maximum shear stress theory:

This theory states that the failure can be assumed to occur when the maximum shear stress in the complex stress system is equal to the value of maximum shear stress in simple tension.

The criterion for the failure may be established as given below :



For a simple tension case

$$\sigma_\theta = \sigma_y \sin^2\theta$$

$$\tau_\theta = \frac{1}{2}\sigma_y \sin 2\theta$$

$$\tau_\theta\big|_{max^m} = \frac{1}{2}\sigma_y \quad \text{or}$$

$$\tau_{max^m} = \frac{1}{2}\sigma_{yp}$$

whereas for the two dimentional complex stress system

$$\tau_{max^m} = \left(\frac{\sigma_1 - \sigma_2}{2}\right)$$

where σ_1 = maximum principle stress

σ_2 = minimum principal stress

$$\text{so} \quad \frac{\sigma_1 - \sigma_2}{2} = \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2 xy}$$

$$\frac{\sigma_1 - \sigma_2}{2} = \frac{1}{2}\sigma_{yp} \Rightarrow \sigma_1 - \sigma_2 = \sigma_y$$

$$\Rightarrow \sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2 xy} = \sigma_{yp}$$

becomes the criterion for the failure.

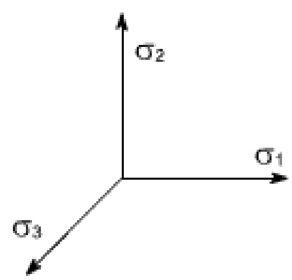
(c) Maximum Principal strain theory:

This Theory assumes that failure occurs when the maximum strain for a complex state of stress system becomes equals to the strain at yield point in the tensile test for the three dimensional complex state of stress system.

For a 3 - dimensional state of stress system the total strain energy U_t per unit volume in equal to the total work done by the system and given by the equation

$$U_t = 1/2\sigma_1 \in_1 + 1/2\sigma_2 \in_2 + 1/2\sigma_3 \in_3$$

substituting the values of $\in_1, \in_2$ and $\in_3$



$$\in_1 = \frac{1}{E}\left[\sigma_1 - \gamma(\sigma_2 + \sigma_3)\right]$$

$$\in_2 = \frac{1}{E}\left[\sigma_2 - \gamma(\sigma_1 + \sigma_3)\right]$$

$$\in_3 = \frac{1}{E}\left[\sigma_3 - \gamma(\sigma_1 + \sigma_2)\right]$$

Thus, the failure criterion becomes

$$\left(\frac{\sigma_1}{E} - \gamma\frac{\sigma_2}{E} - \gamma\frac{\sigma_3}{E}\right) = \frac{\sigma_{yp}}{E}$$

or

$$\boxed{\sigma_1 - \gamma\sigma_2 - \gamma\sigma_3 = \sigma_{yp}}$$

(d) Total strain energy per unit volume theory:

The theory assumes that the failure occurs when the total strain energy for a complex state of stress system is equal to that at the yield point a tensile test.

Therefore, the failure criterion becomes

$$\frac{1}{2E}\left[\sigma_1^2+\sigma_2^2+\sigma_3^2-2\gamma(\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_3\sigma_1)\right]=\frac{\sigma_{yp}^2}{2E}$$
$$\sigma_1^2+\sigma_2^2+\sigma_3^2-2\gamma(\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_3\sigma_1)=\sigma_{yp}^2$$

It may be noted that this theory gives fair by good results for ductile materials.

(e) Maximum shear strain energy per unit volume theory:

This theory states that the failure occurs when the maximum shear strain energy component for the complex state of stress system is equal to that at the yield point in the tensile test.

Hence the criterion for the failure becomes

$$\frac{1}{12G}\left[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2\right]=\frac{\sigma_{yp}^2}{6G}$$

Where G = shear modulus of regidity

$$\left[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2\right]=2\sigma_{yp}^2$$

As we know that a general state of stress can be broken into two components i.e,

(i) Hydrostatic state of stress (the strain energy associated with the hydrostatic state of stress is known as the volumetric strain energy)

(ii) Distortional or Deviator state of stress (The strain energy due to this is known as the shear strain energy) As we know that the strain energy due to distortion is given as

$$U_{distortion}=\frac{1}{12G}\left[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2\right]$$

This is the distortion strain energy for a complex state of stress, this is to be equaled to the maximum distortion energy in the simple tension test. In order to get we may assume that one of the principal stress say (σ_1) reaches the yield point (σ_{yp}) of the material.

Thus, putting in above equation $\sigma_2 = \sigma_3 = 0$

we get distortion energy for the simple test

i.e

$$U_d=\frac{2\sigma_1^2}{12G}$$

Futher $\sigma_1 = \sigma_{yp}$

Thus, $\boxed{U_d=\frac{\sigma_{yp}^2}{6G}}$ for a simple tension test.

